Bergman Measures

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CHAPTER 1

Introduction

Let Ω be a nonempty subset of the complex plane $\mathbb{C}$. By $H(\Omega)$ we shall mean the set of all analytic functions on Ω.

There are several additional structures that can be applied to this set. We can define addition and scalar multiplication on $H(\Omega)$ in the obvious fashion: for f, g in $H(\Omega)$ and $\lambda \in \mathbb{C}$ define functions $f+g$ and λf by

$$\begin{aligned} \big(f+g\big)(z) &= f(z)+g(z); \\ \big(\lambda f\big)(z) &= \lambda \cdot f(z) \qquad (z \in \Omega). \end{aligned}$$

Thus, $H(\Omega)$ becomes a linear space over the field of complex numbers.

Next, we can define a topology on $H(\Omega)$ as follows. A sequence f_1, f_2, ... of functions in $H(\Omega)$ is said to converge *locally uniformly* to a function $f \in H(\Omega)$ if, for every compact subset K of Ω, the sequence f_1, f_2, ... converges to f uniformly on K. It can be shown that there exists a metric d on $H(\Omega)$ such that

$$f_n \xrightarrow{d} f \qquad \text{if and only if} \qquad f_n \to f \text{ locally uniformly.}$$

Furthermore, the space $H(\Omega)$, when endowed with this metric, becomes a complete metric space.

When people speak of $H(\Omega)$, they usually view this set as a linear space with the topology defined above. There are, however, other possible topologies, and this is where things start to get interesting.

Consider the space of all functions in $H(\Omega)$ that are square-integrable with respect to the Lebesgue measure. This space is usually denoted by $A(\Omega)$. On this space we have two topologies: the topology of locally uniform convergence, inherited from $H(\Omega)$, and the so-called L^2-topology, generated by the norm

$$\|f\| = \Big(\int_\Omega |f|^2 \, d\lambda\Big)^{1/2}.$$

(Here λ denotes the Lebesgue measure on Ω.)

It turns out that the L^2-topology and the topology of locally-uniform convergence are related. The key to this is the following rather amazing inequality, or rather set of inequalities:

> for every compact subset K of Ω there exists a constant $C \in [0, \infty)$ such that
>
> $$\big|f(z)\big|^2 \leq C \cdot \int_\Omega |f|^2 \, d\lambda \qquad \text{for all } f \in H(\Omega).$$

This means that every sequence in $A(\Omega)$ that converges in the L^2-topology also converges locally uniformly. This phenomenon is typical of analytic functions. Nothing even remotely like this is true in the general theory of L^2-spaces.

One can prove that the space $A(\Omega)$, when viewed with the L^2-norm, is a Hilbert space. (In fact, this paper contains just such a proof.) Hilbert spaces of analytic, square-integrable functions are called *Bergman spaces*, after S. Bergman, who was one of the first to make a detailed study of such spaces. Bergman spaces turn up in various branches of mathematics. Historically they have been used to attack problems related to conformal mapping, certain extremal problems, and differential equations. For instance, see [**3**], and [**4**]. The focus in the study of Bergman spaces appears to have shifted away from applied mathematics into operator theory, see [**1**] for a fairly recent example.

In this paper we are mostly interested in measures that are like the Lebesgue measure in that they satisfy inequalities similar to the above. To honour Bergman (and, to be honest, more importantly to make this paper a bit more readable) I will call these measures *Bergman measures*. It should be noted that this is *not* a term that is used in the literature.

CHAPTER 2

Examples of Bergman Measures

In this chapter, we define Bergman measures. We prove a simple lemma that enables us to construct a lot of these measures.

DEFINITION. Let Ω be a nonempty open subset of the complex plane $\mathbb{C}$. Let μ be a measure on the Borel-σ-algebra of Ω. Let us call the measure μ a *Bergman measure* if the following two conditions are satisfied:

(i) $\mu(E) < \infty$ for every compact subset E of Ω;

(ii) for every compact subset E of Ω there exists a constant $C \in [0, \infty)$ such that

$$\left|f(z)\right|^2 \leq C \cdot \int_\Omega |f|^2 \, d\mu \qquad \text{for all } z \in E \text{ and every } f \in H(\Omega).$$

A measure that satisfies (i) (but not necessarily (ii)) is sometimes called a Radon measure or a Borel measure. In this paper there are several proofs that certain measures are Bergman measures. Each such proof contains two parts: a part that proves (i), which is usually very trivial, and the interesting part, which proves (ii).

It is possible, in theory, to study measures that satisfy (ii) but not (i). In practice however nearly every measure that is encountered already satisfies (i), therefore it does not hurt incorporating it into the definition. Also the theorems become slightly better-looking this way.

There are a lot of Bergman measures. For example, the Lebesgue measure is one. To see that we need the following crucial lemma. The lemma also helps to construct other Bergman measures.

LEMMA 2.1. *Let Ω be an open subset of $\mathbb{C}$. Let $z \in \Omega$, $r \in (0, \infty)$ such that $\overline{\Delta}_r(z) \subset \Omega$. Let $f \in H(\Omega)$. Then*

$$\left|f(z)\right|^2 \leq \frac{1}{\pi r^2} \int_{\overline{\Delta}_r(z)} |f|^2 \, d\lambda.$$

PROOF.

$$\int_{\overline{\Delta}_r(z)} f \, d\lambda = \int_0^r \rho \int_0^{2\pi} f(z + \rho e^{i\theta}) \, d\theta \, d\rho = \int_0^r \rho \cdot 2\pi f(z) \, d\rho = \pi r^2 f(z).$$

Using the Cauchy-Schwarz inequality for integrals, we see that

$$\begin{aligned} \left|\pi r^2 f(z)\right|^2 &= \left| \int_{\overline{\Delta}_r(z)} f \, d\lambda \right|^2 \\ &\leq \int_{\overline{\Delta}_r(z)} |f|^2 \, d\lambda \cdot \int_{\overline{\Delta}_r(z)} 1 \, d\lambda \\ &= \pi r^2 \int_{\overline{\Delta}_r(z)} |f|^2 \, d\lambda. \end{aligned}$$

□

THEOREM 2.2. *Let Ω be a nonempty open subset of $\mathbb{C}$. Then the Lebesgue measure on Ω is a Bergman measure.*

PROOF. Let E be a compact subset of Ω. Clearly, $\lambda(E) < \infty$. Define the number R as follows:

$$R = \inf\bigl\{|z - w| : z \in E, w \in \mathbb{C} \setminus \Omega\bigr\}.$$

Then $R > 0$. (The number R may be infinite in some degenerate cases but this does not affect our argument.) Choose a number $r \in (0, R)$. Then

$$\overline{\Delta}_r(z) \subseteq \Omega \qquad \text{for every } z \in E.$$

Let $z \in E$, $f \in H(\Omega)$. Then

$$\bigl|f(z)\bigr|^2 \leq \frac{1}{\pi r^2} \int_{\overline{\Delta}_r(z)} |f|^2 \, d\lambda.$$

We now make an extraordinarily crude estimate by noting that

$$\int_{\overline{\Delta}_r(z)} |f|^2 \, d\lambda \leq \int_{\Omega} |f|^2 \, d\lambda.$$

Thus,

$$\bigl|f(z)\bigr|^2 \leq \frac{1}{\pi r^2} \int_{\Omega} |f|^2 \, d\lambda.$$

But this is exactly what we need to prove that λ is a Bergman measure. ☐

Before we proceed with more general examples of Bergman measures we need to make a few preparations.

DEFINITION. For any nonempty subset X of the complex plane, define a function

$$d_X \colon \mathbb{C} \to [0, \infty)$$

as follows:

$$d_X(z) = \inf\bigl\{|z - w| : w \in X\bigr\}.$$

The number $d_X(z)$ represents the distance from the point z to the set X, informally speaking.

LEMMA 2.3. *Let X be a nonempty subset of $\mathbb{C}$. Let $z_1 \in \mathbb{C}$, $z_2 \in \mathbb{C}$. Then*

$$\bigl|d_X(z_2) - d_X(z_1)\bigr| \leq |z_2 - z_1|.$$

PROOF. Let $\varepsilon \in (0, \infty)$. There exist $w_1, w_2 \in X$ such that

$$|z_1 - w_1| \leq d_X(z_1) + \varepsilon$$

and

$$|z_2 - w_2| \leq d_X(z_2) + \varepsilon.$$

We then have

$$d_X(z_2) - d_X(z_1) \leq |z_2 - w_1| - |z_1 - w_1| + \varepsilon \leq |z_2 - z_1| + \varepsilon,$$

and

$$d_X(z_1) - d_X(z_2) \leq |z_1 - w_2| - |z_2 - w_2| + \varepsilon \leq |z_1 - z_2| + \varepsilon.$$

It follows that

$$\bigl|d_X(z_2) - d_X(z_1)\bigr| \leq |z_2 - z_1| + \varepsilon$$

for every $\varepsilon > 0$. But this is only possible if $\bigl|d_X(z_2) - d_X(z_1)\bigr| \leq |z_2 - z_1|$. ☐

COROLLARY 2.4. *For every nonempty subset X of $\mathbb{C}$ the function d_X is continuous.*

COROLLARY 2.5. *Let E be a nonempty, bounded subset of $\mathbb{C}$. Let $r \in (0, \infty)$. Let*

$$F = \{z \in \mathbb{C} : d_E(z) \leq r\}.$$

Then F is compact.

PROOF. Because E is bounded, F is bounded. Because d_E is continuous, F is closed. Hence F is compact. □

THEOREM 2.6. *Let Ω be an open subset of $\mathbb{C}$, and let $\omega\colon \Omega \to (0,\infty)$ be a continuous function. Then the measure $\omega\lambda$ is a Bergman measure.*

PROOF. The proof is roughly the same as the proof that the Lebesgue measure is a Bergman measure, except that we now have the weight function ω to deal with. Let E be a compact subset of Ω. Since the function ω is continuous, it is bounded on E, hence $\omega\lambda(E) < \infty$. We are done if we can find a constant $C \in [\,0,\infty)$ such that

$$\big|f(z)\big|^2 \le C \cdot \int_\Omega |f|^2 \, d\omega\lambda \qquad \text{for all } z \in E \text{ and all } f \in H(\Omega).$$

We may assume that E is not empty. Again let

$$R = \inf\big\{|z-w| : z \in E, w \in \mathbb{C} \setminus \Omega\big\}.$$

Choose a number $r \in (0, R)$. Let

$$F = \{z \in \mathbb{C} : d_E(z) \le r\}.$$

Then

$$E \subseteq F \subseteq \Omega.$$

Since the set F is compact (see above) and the function ω is continuous, there exists a number $\delta \in (0,\infty)$ such that

$$\omega(z) \ge \delta \qquad \text{for all } z \in F.$$

Let $z \in E$, and $f \in H(\Omega)$. Then $\overline{\Delta}_r(z) \subseteq F$. Hence

$$\int_{\overline{\Delta}_r(z)} |f|^2 \, d\omega\lambda \ge \delta \int_{\overline{\Delta}_r(z)} |f|^2 \, d\lambda,$$

and

$$\begin{aligned}\big|f(z)\big|^2 &\le \frac{1}{\pi r^2} \int_{\overline{\Delta}_r(z)} |f|^2 \, d\lambda \\ &\le \frac{1}{\delta \pi r^2} \int_{\overline{\Delta}_r(z)} |f|^2 \, d\omega\lambda \\ &\le \frac{1}{\delta \pi r^2} \int_\Omega |f|^2 \, d\omega\lambda.\end{aligned}$$

□

The previous theorem shows that a continuous, strictly positive weight function gives rise to Bergman measures. It turns out that the weight function may even vanish on certain parts of its domain, as long as it is nonzero 'sufficiently often' near the boundary of the domain.

THEOREM 2.7. *Let Ω be a nonempty open subset of $\mathbb{C}$. Let $\omega\colon \Omega \to [\,0,\infty)$ be a measurable, locally integrable function. Let E_0 be a compact subset of Ω such that the function ω is continuous on $\Omega \setminus E_0$ and*

$$\omega(z) > 0 \qquad \text{for all } z \in \Omega \setminus E_0.$$

Then $\omega\lambda$ is a Bergman measure on Ω.

PROOF. Let E be a compact subset of Ω. Since the function ω is locally integrable, $\omega\lambda(E) < \infty$. Next, we try to find a constant $C \in [\,0,\infty)$ such that

$$\big|f(z)\big|^2 \le C \cdot \int_\Omega |f|^2 \, d\omega\lambda \qquad \text{for all } z \in E \text{ and all } f \in H(\Omega).$$

Consider the set $E \cup E_0$. This set is also a compact subset of Ω. If we can find a constant for the union of E and E_0, then certainly that constant will also do for E itself. In other words: we may assume that

$$E_0 \subseteq E.$$

Furthermore, we may also assume that the set E is not empty. Once again, let

$$R = \inf\bigl\{|z - w| : z \in E, w \in \mathbb{C} \setminus \Omega\bigr\}.$$

Choose a number $r \in (0, R)$. Define the set F as follows:

$$F = \left\{z \in \mathbb{C} : d_E(z) \leq \frac{1}{2}r\right\}.$$

Clearly, the set F is compact, $E \subseteq F$, and

$$\partial F = \left\{z \in \mathbb{C} : d_E(z) = \frac{1}{2}r\right\}.$$

Let

$$F' = \left\{z \in \mathbb{C} : \frac{1}{4}r \leq d_E(z) \leq \frac{3}{4}r\right\}.$$

The set F' is also compact, and $\partial F \subseteq F'$. Furthermore, because $E_0 \subseteq E$,

$$F' \subseteq \Omega \setminus E_0.$$

Since the function ω is continuous on $\Omega \setminus E_0$, there exists a number $\delta \in (0, \infty)$ such that

$$\omega(z) \geq \delta \qquad \text{for all } z \in F'.$$

We have

$$\overline{\Delta}_{\frac{1}{4}r}(z) \subseteq F' \qquad \text{for all } z \in \partial F.$$

Let $z \in E$, $f \in H(\Omega)$. The point z is also a member of the set F. From the maximum principle, applied to the function f^2, it follows that there exists a point $w \in \partial F$ such that

$$\bigl|f(z)\bigr|^2 \leq \bigl|f(w)\bigr|^2.$$

Now,

$$\int_{\overline{\Delta}_{\frac{1}{4}r}(w)} |f|^2 \, d\omega\lambda \geq \delta \int_{\overline{\Delta}_{\frac{1}{4}r}(w)} |f|^2 \, d\lambda,$$

and

$$\begin{aligned}
\bigl|f(w)\bigr|^2 &\leq \frac{16}{\pi r^2} \int_{\overline{\Delta}_{\frac{1}{4}r}(w)} |f|^2 \, d\lambda \\
&\leq \frac{16}{\delta\pi r^2} \int_{\overline{\Delta}_{\frac{1}{4}r}(w)} |f|^2 \, d\omega\lambda \\
&\leq \frac{16}{\delta\pi r^2} \int_\Omega |f|^2 \, d\omega\lambda,
\end{aligned}$$

Putting everything together, we see that

$$\bigl|f(z)\bigr|^2 \leq \frac{16}{\delta\pi r^2} \int_\Omega |f|^2 \, d\omega\lambda. \qquad \square$$

The above proof rests on two observations. For every compact subset of Ω one can construct a 'ring' inside Ω around that subset that is also compact and on which the weight function is well behaved. Using the maximum principle one can then ignore what goes on in the original subset, and inspect what goes on in the ring instead.

We can take this idea one step further.

THEOREM 2.8. *Let Ω be a nonempty open subset of $\mathbb{C}$. Let $\omega\colon \Omega \to [0,\infty)$ be a measurable, locally integrable function that is such that there exists a sequence E_0, E_1, E_2, ... of subsets of Ω and a sequence a_0, a_1, a_2, ... of numbers such that*

- *E_n is compact and not empty for all $n \in \mathbb{N}$,*
- *$E_n \subseteq E_{n+1}^\circ$ for all $n \in \mathbb{N}$,*
- *$\bigcup_{n\in\mathbb{N}} E_n = \Omega$,*
- *$a_n > 0$ for all $n \in \mathbb{N}$,*
- *$\omega(z) \geq a_n$ for all $n \in \mathbb{N}$ and all $z \in E_{2n+1}^\circ \setminus E_{2n}$.*

Then the measure $\omega\lambda$ is a Bergman measure.

PROOF. Let E be a compact subset of Ω. Because the function ω is locally integrable, $\omega\lambda(E) < \infty$. There exists a number $n \in \mathbb{N}$ such that

$$E \subseteq E_{2n}^\circ.$$

Let

$$r = \inf\bigl\{|z-w| : z \in E_{2n}, w \in \mathbb{C} \setminus E_{2n+1}^\circ\bigr\}.$$

Because $E_{2n} \subseteq E_{2n+1}^\circ$, we have $r > 0$. Also, $r < \infty$, because the set E_{2n} is not empty. Let

$$F = \left\{ z \in \mathbb{C} : d_{E_{2n}}(z) \leq \frac{1}{2} r \right\}.$$

Then

$$E \subseteq F$$

and

$$\overline{\Delta}_{\frac{1}{4}r}(w) \subseteq E_{2n+1}^\circ \setminus E_{2n} \qquad \text{for all } w \in \partial F.$$

Let $z \in E$, $f \in H(\Omega)$. Then $z \in F$. From the maximum principle, applied to the function f^2, it follows that there exists a $w \in \partial F$ such that $\bigl|f(z)\bigr|^2 \leq \bigl|f(w)\bigr|^2$. Then

$$\begin{aligned}
\bigl|f(z)\bigr|^2 &\leq \bigl|f(w)\bigr|^2 \\
&\leq \frac{16}{\pi r^2} \int_{\Delta_{\frac{1}{4}r}(w)} |f|^2 \, d\lambda \\
&\leq \frac{16}{a_n \pi r^2} \int_{\Delta_{\frac{1}{4}r}(w)} |f|^2 \, d\omega\lambda \\
&\leq \frac{16}{a_n \pi r^2} \int_{\Omega} |f|^2 \, d\omega\lambda.
\end{aligned}$$

□

The idea behind this is that we slice our set Ω into infinitely many compact rings that together make up our original set. If the weight function stays a certain distance away from zero on every other ring, we get a Bergman measure, even if the function vanishes everywhere else.

EXAMPLE. Define a function ω on Δ as follows:

$$\omega(z) = \begin{cases} 1 & \text{if } \left[\frac{1}{1-|z|}\right] \text{ is even;} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\omega\lambda$ is a Bergman measure.

CHAPTER 3

Properties of Bergman Measures

In the previous chapter we have collected a large number of Bergman measures. Now we will try to prove some theorems about these measures.

Thus far, every measure we have inspected has turned out to be a Bergman measure. One may wonder whether there are in fact measures that are *not* Bergman measures. In fact, the zero measure is not a Bergman measure. This follows at once from the following.

LEMMA 3.1. *Let Ω be a nonempty open subset of $\mathbb{C}$, and let μ be a Bergman measure on Ω. Then*

$$\mu(\Omega) > 0.$$

PROOF. Choose any point $z \in \Omega$. Since the set $\{z\}$ is compact, there exists a constant $C \in [\,0, \infty)$ such that

$$1 \leq C \cdot \int_\Omega 1\, d\mu = C \cdot \mu(\Omega).$$

But this is only possible if $\mu(\Omega) > 0$. □

All Bergman measures we have constructed in the previous chapter live near the boundary of their domain (to use a dodgy metaphor.) It turns out that this is a fundamental property that is shared by all Bergman measures. This is expressed by the following theorem, of which the above lemma is a special case. While the above lemma gives us a rather trivial example of a non-Bergman measure, the next theorem will provide us with very many, very nontrivial examples.

THEOREM 3.2. *Let Ω be a nonempty open subset of $\mathbb{C}$. Let μ be a Bergman measure on Ω. Let E be a compact subset of Ω. Then*

$$\mu(\Omega \setminus E) > 0.$$

PROOF. The case when the set E is empty is taken care of by the previous lemma. So we can assume that E is not empty.

Suppose, $\mu(\Omega \setminus E) = 0$.

Choose a point $z_0 \in \Omega \setminus E$. Let

$$R = \sup\bigl\{|z - z_0| : z \in E\bigr\}.$$

Since the set E is compact and not empty, $0 < R < \infty$. Also, because E is compact and Ω is open, there exists a point $z_1 \in \Omega$ such that

$$|z_1 - z_0| > R.$$

The set $\{z_1\}$ is compact. There exists a constant $C \in [\,0, \infty)$ such that

$$\bigl|f(z_1)\bigr|^2 \leq C \cdot \int_\Omega |f|^2\, d\mu \qquad \text{for all } f \in H(\Omega).$$

Let $n \in \mathbb{N}$. Applying the previous inequality to the function

$$z \mapsto \left(\frac{z - z_0}{R}\right)^n,$$

we see that

$$\begin{aligned}
\left|\frac{z_1 - z_0}{R}\right|^{2n} &\leq C \cdot \int_\Omega \left|\frac{z - z_0}{R}\right|^{2n} d\mu(z) \\
&= C \cdot \left(\int_E \left|\frac{z - z_0}{R}\right|^{2n} d\mu(z) + \int_{\Omega\setminus E} \left|\frac{z - z_0}{R}\right|^{2n} d\mu(z)\right) \\
&= C \cdot \int_E \left|\frac{z - z_0}{R}\right|^{2n} d\mu(z) \\
&\leq C \cdot \mu(E).
\end{aligned}$$

Let $x = \left|(z_1 - z_0)/R\right|^2$. Since $\mu(E) < \infty$, the sequence x, x^2, x^3, ... is bounded. But that can't possibly be true, because $x > 1$.

The assumption that $\mu(\Omega\setminus E) = 0$ leads to a contradiction. So $\mu(\Omega\setminus E) > 0$. □

The central idea in the proof is that we can construct a sequence of functions that are uniformly bounded on a compact subset, but that 'explode' on a certain point outside of that subset.

Another application of this idea is the following.

Theorem 3.3. *Let Ω be a nonempty open subset of $\mathbb{C}$. Let μ be a Bergman measure on Ω. Let $h \in H(\Omega)$. Let the set*

$$X = \left\{z \in \Omega : \left|h(z)\right| > 1\right\}$$

be such that

$$\mu(\Omega \setminus X) < \infty$$

and

$$\mu(X) = 0.$$

Then X is empty.

Proof. Let $z \in \Omega$. We are done if we can show that $\left|h(z)\right| \leq 1$. Choose a constant $C \in [0, \infty)$ such that

$$|f(z)|^2 \leq C \cdot \int_\Omega |f|^2 \, d\mu \qquad \text{for all } f \in H(\Omega).$$

Let $n \in \mathbb{N}$. Applying the previous inequality to the function h^n, we have

$$\begin{aligned}
\left|h(z)\right|^{2n} &\leq C \cdot \int_\Omega |h^n|^2 \, d\mu \\
&= C \cdot \int_{\Omega\setminus X} |h^n|^2 \, d\mu \\
&\leq C \cdot \int_{\Omega\setminus X} 1 \, d\mu \\
&= C \cdot \mu(\Omega \setminus X).
\end{aligned}$$

Since $\mu(\Omega \setminus X) < \infty$, the sequence $n \mapsto h^n(z)$ is bounded. But that is only possible if $\left|h(z)\right| \leq 1$. □

The theorem above can help us show that certain subsets cannot have measure zero, by constructing appropriate functions.

Example. Let μ be a Bergman measure on the unit disc. To simplify matters we assume that μ is finite.

Let $r \in (0, 1)$.

Define a function f on the unit disc by

$$f(z) = \frac{1+z}{1-z}.$$

Let D_1 be the disc with centre $1-r$ and radius r. Then D_1 touches the unit circle at 1. The function f maps D_1 onto the half-plane $\{w \in \mathbb{C} : \Re w > 1/r - 1\}$. Define the function $h_1 \in H(\Delta)$ by

$$h_1(z) = \exp\bigl(f(z) - (1/r - 1)\bigr).$$

It is obvious that, for every $z \in \Delta$,

$$\bigl|h_1(z)\bigr| > 1 \iff z \in D_1.$$

It follows that $\mu(D_1)$ must be nonzero.

Consider the sector $D_2 = \{z \in D_1 : -\pi/4 < \arg(1-z) < \pi/4\}$. Using elementary calculations one can show that the reverse image under f^2 of the half-plane $\bigl\{w \in \mathbb{C} : \Re w > (1/r-1)^2\bigr\}$, which is a nonempty set, is wholly contained within D_2. Using the function

$$h_2(z) = \exp\bigl(f^2(z) - (1/r-1)^2\bigr)$$

we see that $\mu(D_2)$ must be nonzero.

In fact we can go one step further. Let D_3 be the sector $\{z \in D_1 : -\pi/6 < \arg(1-z) < \pi/6\}$. Then the nonempty reverse image under f^3 of the half-plane $\bigl\{w \in \mathbb{C} : \Re w > (1/r-1)^3\bigr\}$ is wholly contained within D_3. Again we see that $\mu(D_3) > 0$.

We see that we can find sectors that have nonzero measure with ever smaller angles, up to $\pi/3$, simply by increasing an exponent. Unfortunately this procedure breaks down at exponent 4. The reverse image under f^4 of a half-plane is no longer contained within a sector with angle $\pi/4$.

It is still true, however, that the measure of each sector at the unit circle must be nonzero. One can construct an entire function that is bounded, except on a horizontal strip (see [**2**]). Using appropriate variations of this function one can show that each sector at the unit circle must have nonzero measure. Unfortunately the construction of this function is far from elementary and far beyond the scope of this paper.

Next, we prove two generalisations of lemma 3.1 and theorem 3.2.

LEMMA 3.4. *Let Ω be a nonempty subset of $\mathbb{C}$. Let μ be a Bergman measure on Ω. Let $g \in H(\Omega)$ be such that*

$$\int_\Omega |g|^2 \, d\mu = 0.$$

Then $g = 0$.

PROOF. Let $z \in \Omega$. The set $\{z\}$ is compact, hence there exists a $C \in [0, \infty)$ such that

$$\bigl|g(z)\bigr|^2 \le C \cdot \int_\Omega |g|^2 \, d\mu$$

But that means that $g(z)$ must be zero. □

THEOREM 3.5. *Let Ω be a nonempty, connected, open subset of $\mathbb{C}$. Let μ be a Bergman measure on Ω. Let E be a compact subset of Ω. Let $g \in H(\Omega)$ be such that*

$$\int_{\Omega\setminus E} |g|^2 \, d\mu = 0.$$

Then $g = 0$.

The proof is similar to that of theorem 3.2, with a slight twist.

PROOF. We have just proved this theorem in case the set E is empty. So now we may assume that E is not empty. Let $z_0 \in \Omega \setminus E$. Let

$$R = \sup\{|z - z_0| : z \in E\}.$$

Then $0 < R < \infty$.

Let $z \in \Omega$ be such that $|z - z_0| > R$. The set $\{z\}$ is compact. Hence, there exists a constant $C \in [0, \infty)$ such that

$$\big|f(z)\big|^2 \le C \cdot \int_\Omega |f|^2 \, d\mu \qquad \text{for all } f \in H(\Omega).$$

Let $n \in \mathbb{N}$. Then

$$\begin{aligned} \left|\left(\frac{z - z_0}{R}\right)^n \cdot g(z)\right|^2 &\le C \cdot \int_\Omega \left|\left(\frac{w - z_0}{R}\right)^n \cdot g(w)\right|^2 \, d\mu(w) \\ &= C \cdot \int_E \left|\left(\frac{w - z_0}{R}\right)^n \cdot g(w)\right|^2 \, d\mu(w) \\ &\le C \cdot \mu(E) \cdot \int_E |g|^2 \, d\mu(w) \end{aligned}$$

We see that

$$\left|\left(\frac{z - z_0}{R}\right)^n \cdot g(z)\right|^2 \le k$$

for each $n \in \mathbb{N}$, where k is some finite constant. But that is only possible if $g(z) = 0$.

The function g vanishes on the set $\{z \in \Omega : |z - z_0| > R\}$. This set is open and not empty. Since Ω is connected, g must vanish everywhere. □

The theorem is somewhat blemished by the fact that we now suddenly require the set Ω to be connected. In fact the theorem is true for *any* open subset of $\mathbb{C}$. To see this we need the following lemma.

LEMMA 3.6. *Let Ω be a nonempty open subset of $\mathbb{C}$. Let μ be a Bergman measure on Ω. Let Ω' be a component of Ω. Then the restriction of μ to Ω' is a Bergman measure on Ω'.*

PROOF. Let μ' be the restriction of μ to Ω'. Let E be a compact subset of Ω'. Obviously, $\mu'(E) < \infty$. There exists a constant $C \in [0, \infty)$ such that

$$\big|f(z)\big|^2 \le C \cdot \int_\Omega |f|^2 \, d\mu \qquad \text{for all } f \in H(\Omega).$$

Let $g \in H(\Omega')$. Define a function $\tilde{g}$ on Ω as follows:

$$\tilde{g}(z) = \begin{cases} g(z) & \text{if } z \in \Omega'; \\ 0 & \text{otherwise.} \end{cases} \qquad (z \in \Omega)$$

Then $\tilde{g} \in H(\Omega)$. Let $z \in \Omega'$. Then

$$\big|g(z)\big|^2 = \big|\tilde{g}(z)\big|^2 \le C \cdot \int_\Omega |\tilde{g}|^2 \, d\mu = C \cdot \int_{\Omega'} |g|^2 \, d\mu'.$$ □

THEOREM 3.7. *Let Ω be a nonempty open subset of $\mathbb{C}$. Let μ be a Bergman measure on Ω. Let E be a compact subset of Ω. Let $g \in H(\Omega)$ be such that*

$$\int_{\Omega \setminus E} |g|^2 \, d\mu = 0$$

Then $g = 0$.

PROOF. Let $z \in \Omega$. Let G be the component of Ω that contains z. The restriction of μ to G is Bergman measure on G. Since $E \cap G$ is compact, and

$$\int_{(\Omega \setminus E) \cap G} |g|^2 \, d\mu = 0$$

it follows that $g(z) = 0$. □

The following result is much deeper.

THEOREM 3.8. *Let Ω be a nonempty open subset of $\mathbb{C}$. Let μ be a Bergman measure on Ω. Let E be a compact subset of Ω. Then there exists a constant $c \in [0, 1)$ such that*

$$\int_E |f|^2 \, d\mu \leq c \cdot \int_\Omega |f|^2 \, d\mu \qquad \text{for all } f \in H(\Omega).$$

The following proof can be found in [**1**].

PROOF. Suppose, such a constant does not exist.

Then there exists a sequence of functions f_1, f_2, f_3, … in $H(\Omega)$ such that

$$\int_E |f_n|^2 \, d\mu > \left(1 - \frac{1}{n}\right) \int_\Omega |f_n|^2 \, d\mu \qquad \text{for each } n \in \mathbb{N}^*.$$

Then, for each $n \in \mathbb{N}^*$,

$$0 < \int_\Omega |f_n|^2 \, d\mu < \infty.$$

Define another sequence g_1, g_2, g_3, … as follows:

$$g_n = \left(\int_\Omega |f_n|^2 \, d\mu\right)^{-\frac{1}{2}} \cdot f_n \qquad (n \in \mathbb{N}^*.)$$

Let $n \in \mathbb{N}^*$. Then

$$\int_\Omega |g_n|^2 \, d\mu = 1,$$

and

$$\int_E |g_n|^2 > 1 - \frac{1}{n}.$$

Hence

$$\int_{\Omega \setminus E} |g_n|^2 < \frac{1}{n}.$$

It follows that

$$\lim_{n \to \infty} \int_E |g_n|^2 \, d\mu = 1,$$

and

$$\lim_{n \to \infty} \int_{\Omega \setminus E} |g_n|^2 \, d\mu = 0.$$

The sequence g_1, g_2, g_3, … is locally uniformly bounded, and hence has a locally uniformly converging subsequence g_{n_1}, g_{n_2}, g_{n_3}, …. Let

$$g = \lim_{k \to \infty} g_{n_k}.$$

Then $g \in H(\Omega)$. By dominated convergence,

$$\int_E |g|^2 \, d\mu = \lim_{k \to \infty} \int_E |g_{n_k}|^2 \, d\mu = 1,$$

but Fatou's lemma says that

$$\int_{\Omega \setminus E} |g|^2 \, d\mu = 0.$$

It follows that $g = 0$, and hence $0 = 1$. But that is absurd. □

The above theorem states that the integral of the square of a function over a compact subset always stays a certain distance away from the overall integral.

The following result is presented as a corollary, but it is in fact equivalent to the above. (In fact in [**1**], the above proof is used in a proof of 3.9.)

COROLLARY 3.9. *Let Ω be a nonempty open subset of $\mathbb{C}$. Let μ be a Bergman measure on Ω. Let E be a compact subset of Ω. Then there exists a constant $C \in [1, \infty)$ such that*

$$\int_\Omega |f|^2 \, d\mu \leq C \cdot \int_{\Omega \setminus E} |f|^2 \, d\mu \qquad \text{for all } f \in H(\Omega).$$

PROOF. Let $c \in [0, 1)$ be such that $\int_E |f|^2 \, d\mu \leq c \cdot \int_\Omega |f|^2 \, d\mu$ for all $f \in H(\Omega)$. Let $f \in H(\Omega)$. Then

$$\int_\Omega |f|^2 \, d\mu = \int_E |f|^2 \, d\mu + \int_{\Omega \setminus E} |f|^2 \, d\mu \leq c \cdot \int_\Omega |f|^2 \, d\mu + \int_{\Omega \setminus E} |f|^2 \, d\mu.$$

Hence,

$$\int_\Omega |f|^2 \leq \frac{1}{1-c} \int_{\Omega \setminus E} |f|^2 \, d\mu.$$

□

The above can be used to construct more Bergman measures. In fact, it is now easy to prove the following.

THEOREM 3.10. *Let Ω be a nonempty subset of $\mathbb{C}$. Let μ be a Bergman measure on Ω. Let ν be a measure on Ω that is such that there exists a compact subset E_0 of Ω such that*

$$\nu(E_0) < \infty$$

and

$$\nu(X) = \mu(X) \qquad \text{for every measurable } X \subseteq \Omega \setminus E_0.$$

Then ν is a Bergman measure. In fact, there exists a constant $C \in [0, \infty)$ such that

$$\int_\Omega |f|^2 \, d\mu \leq C \int_\Omega |f|^2 \, d\nu \qquad \text{for all } f \in H(\Omega).$$

PROOF. Because $\nu(E_0) < \infty$ and μ and ν are the same outside of E_0, the measure ν has the property that $\nu(E) < \infty$ for every compact $E \subseteq \Omega$.

By corollary 3.9, there exists a $C \in [1, \infty)$ with

$$\int_\Omega |f|^2 \, d\mu \leq C \cdot \int_{\Omega \setminus E_0} |f| \, d\mu \qquad \text{for all } f \in H(\Omega).$$

Choose such a C. Let $f \in H(\Omega)$ Then, again because the measures μ and ν agree outside E_0,

$$\int_{\Omega \setminus E_0} |f|^2 \, d\mu = \int_{\Omega \setminus E_0} |f|^2 \, d\nu,$$

hence

$$\begin{aligned} \int_\Omega |f|^2 \, d\mu &\leq C \cdot \int_{\Omega \setminus E_0} |f|^2 \, d\mu \\ &\leq C \cdot \int_{\Omega \setminus E_0} |f|^2 \, d\nu \\ &\leq C \cdot \int_\Omega |f|^2 \, d\nu. \end{aligned}$$

Since μ is a Bergman measure it follows easily that ν is one too. □

Note that the above theorem yields another, completely different, proof of theorem 2.7.

CHAPTER 4

The Bergman Space and the Bergman Kernel

Definition. Let μ be a Bergman measure on a nonempty open set Ω. We define the linear space $A(\Omega,\mu)$ as follows:

$$A(\Omega,\mu) = \left\{ f \in H(\Omega) : \int_\Omega |f|^2 \, d\mu < \infty \right\}.$$

In other words, $A(\Omega,\mu) = H(\Omega) \cap \mathcal{L}^2(\mu)$.

For $f, g \in A(\Omega,\mu)$, define

$$\langle f, g\rangle = \int_\Omega f \cdot \overline{g} \, d\mu.$$

We have already seen that a function f for which $\langle f, f\rangle = 0$ must be zero, so $\langle\ ,\ \rangle$ is an inner product on $A(\Omega,\mu)$. This enables us to define a norm on that space. For $f \in A(\Omega,\mu)$, define

$$\|f\| = \sqrt{\langle f, f\rangle}.$$

When we talk about things like 'converging', 'completeness', etc. in $A(\Omega,\mu)$ we will view the space with the norm defined above, unless explicitly stated otherwise.

Theorem 4.1. *Let Ω be a nonempty open subset of $\mathbb{C}$, and μ be a Bergman measure on Ω. Then the space $A(\Omega,\mu)$ is complete.*

Proof. Let f_1, f_2, ... be a Cauchy sequence in $A(\Omega,\mu)$.

Let E be a compact subset of Ω. Choose a number $C \in (0,\infty)$ such that

$$\big|g(z)\big|^2 \le C \cdot \int_\Omega |g|^2 \, d\mu \qquad \text{for all } z \in E \text{ and all } g \in H(\Omega).$$

Let $\varepsilon > 0$. There exists an $N \in \mathbb{N}^*$ such that

$$\|f_n - f_m\| \le \frac{\varepsilon}{\sqrt{C}} \qquad \text{for all } n, m \in \mathbb{N}^* \text{ with } n > N, m > N.$$

For each $z \in E$, and every $n, m \in \mathbb{N}^*$ such that $n > N$, $m > N$,

$$\big|f_n(z) - f_m(z)\big| \le \sqrt{C} \cdot \|f_n - f_m\| \le \epsilon.$$

From this it follows that there exists a function $f \in H(\Omega)$ such that

$$f_n \to f \qquad \text{locally uniformly.}$$

On the other hand, by the Riesz-Fischer theorem, there exists a function $\tilde{f} \in \mathcal{L}^2(\mu)$ such that

$$f_n(z) \to \tilde{f}(z) \qquad \text{for almost every } z \in \Omega$$

and

$$\lim_{n\to\infty} \|f_n - \tilde{f}\| = 0.$$

But then $f = \tilde{f}$ almost everywhere. We conclude that $f \in A(\Omega,\mu)$ and that the sequence f_1, f_2, ... converges to f in the $\mathcal{L}^2$-sense. □

The space $A(\Omega,\mu)$ is called the *Bergman space* associated with Ω and μ. The above theorem states that every Bergman space is a Hilbert space.

THEOREM 4.2. *Let Ω be a nonempty open subset of $\mathbb{C}$ and let μ be a Bergman measure on Ω. Then for every $z \in \Omega$ there exists a unique function $K_z \in A(\Omega, \mu)$ such that*

$$f(z) = \int_\Omega f \cdot \overline{K_z}\, d\mu \qquad \text{for every } f \in A(\Omega, \mu).$$

PROOF. Let $z \in \Omega$. Define a linear function

$$\delta_z \colon A(\Omega, \mu) \to \mathbb{C}$$

as follows:

$$\delta_z(f) = f(z).$$

Since the set $\{z\}$ is compact there exists a number $C \in [0, \infty)$ such that

$$\big|f(z)\big|^2 \leq C \cdot \int_\Omega |f|^2\, d\mu$$

for all $f \in A(\Omega, \mu)$. It follows that

$$\big|\delta_z(f)\big| \leq \sqrt{C} \cdot \|f\|$$

for every $f \in A(\Omega, \mu)$. In other words, the function δ_z is continuous. We now use the fact that $A(\Omega, \mu)$ is a Hilbert space. According to the Riesz-Frechet representation theorem there exists a unique function $K_z \in A(\Omega, \mu)$ such that $\delta_z(f) = \langle f, K_z \rangle$ for all $f \in A(\Omega, \mu)$. In other words

$$f(z) = \int_\Omega f \cdot \overline{K_z}\, d\mu$$

for all $f \in A(\Omega, \mu)$. But this is exactly what we wanted to prove. □

COROLLARY 4.3. *Let Ω be a nonempty open subset of $\mathbb{C}$ and let μ be a Bergman measure on Ω. Then there exists a unique function*

$$K \colon \Omega \times \Omega \to \mathbb{C}$$

such that for every $z \in \Omega$, the function $w \mapsto K(w, z)$ is an element of $A(\Omega, \mu)$, and

$$f(z) = \int_\Omega f(w) \cdot \overline{K(w, z)}\, d\mu(w) \qquad \text{for every } f \in A(\Omega, \mu) \text{ and every } z \in \Omega.$$

PROOF. Define, for $w, z \in \Omega$, $K(w, z) = K_z(w)$, where K_z is as in the previous theorem. It is trivial to see that this function K has the desired properties. □

The function K is called the *Bergman kernel* associated with Ω and μ. It allows one to reconstruct the value for any function in any of its points.

We will continue to denote the function $w \mapsto K(w, z)$ as K_z. Using our inner production notation we have, for $z \in \Omega$, and $f \in A(\Omega, \mu)$,

$$f(z) = \langle f, K_z \rangle.$$

This leads to a few elementary results.

PROPOSITION 4.4. *Let K be the Bergman kernel associated with Ω and μ. Let $w, z \in \Omega$. Then*

$$K(z, w) = \overline{K(w, z)}.$$

PROOF.

$$K(z, w) = K_w(z) = \langle K_w, K_z \rangle = \overline{\langle K_z, K_w \rangle} = \overline{K_z(w)} = \overline{K(w, z)}. \qquad \square$$

PROPOSITION 4.5. *Let K be the Bergman kernel associated with Ω and μ. Let $z \in \Omega$. Then*

$$K(z, z) = \|K_z\|^2.$$

PROOF.
$$K(z,z) = K_z(z) = \langle K_z, K_z \rangle = \|K_z\|^2. \qquad \square$$

COROLLARY 4.6. *For every $z \in \Omega$,*
$$K(z,z) \geq 0.$$

The following proposition tells us exactly how big a function can become at a certain point, compared to its norm. It is a direct consequence of the Cauchy-Schwarz inequality.

PROPOSITION 4.7. *Let K be the Bergman kernel associated with Ω and μ. Let $z \in \Omega$, $f \in A(\Omega, \mu)$. Then*
$$\big|f(z)\big| \leq \sqrt{K(z,z)} \cdot \|f\|.$$
Equality occurs if and only if f is a multiple of K_z.

PROOF.
$$\big|f(z)\big| = \big\langle f, K_z \big\rangle \leq \|f\| \cdot \|K_z\| = \|f\| \cdot \sqrt{K(z,z)}.$$
Should equality occur, then $\langle f, K_z \rangle = \|f\| \cdot \|K_z\|$, but this can happen only if the function f and K_z are linearly dependent. $\square$

PROPOSITION 4.8. *Let K be the Bergman kernel associated with Ω and μ. Let $(h_i)_{i \in I}$ be an orthonormal base of $A(\Omega, \mu)$. Let $w, z \in \Omega$. Then*
$$K(w,z) = \sum_{i \in I} h_i(w) \cdot \overline{h_i(z)}.$$

PROOF.
$$K_z = \sum_{i \in I} \langle K_z, h_i \rangle \cdot h_i = \sum_{i \in I} \overline{\langle h_i, K_z \rangle} \cdot h_i = \sum_{i \in I} \overline{h_i(z)} \cdot h_i. \qquad \square$$

The above proposition tells us that the Bergman kernel can be computed from orthonormal bases. This will now be the focus on our attention.

CHAPTER 5

The Lebesgue Measure on the Unit Disc

We inspect the Lebesgue measure on the unit disc Δ. On the unit disc we have the 'powers of z', $\mathfrak{z}^0$, $\mathfrak{z}^1$, $\mathfrak{z}^2$, ... Clearly every one of these functions is a member of $A(\Delta, \lambda)$.

LEMMA 5.1. *The functions $\mathfrak{z}^0$, $\mathfrak{z}^1$, $\mathfrak{z}^2$, ... form an orthogonal set. Furthermore,*

$$\|\mathfrak{z}^n\| = \sqrt{\frac{\pi}{n+1}} \qquad \textit{for each } n \in \mathbb{N}.$$

PROOF. Let $n, m \in \mathbb{N}$. Then

$$\begin{aligned}
\langle \mathfrak{z}^n, \mathfrak{z}^m \rangle &= \int_\Delta z^n \overline{z^m} \, dz \\
&= \int_0^1 \rho \int_0^{2\pi} (\rho e^{i\theta})^n \overline{(\rho e^{i\theta})^m} \, d\theta \, d\rho \\
&= \int_0^1 \rho \int_0^{2\pi} \rho^{n+m} e^{(n-m)i\theta} \, d\theta \, d\rho \\
&= \begin{cases} \frac{\pi}{n+1} & \text{if } n = m; \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

□

THEOREM 5.2. *Let $f \in A(\Delta, \lambda)$. Let a_0, a_1, a_2, ... be numbers such that*

$$f(z) = \sum_{j=0}^\infty a_j z^j \qquad \textit{for each } z \in \Delta.$$

Let $n \in \mathbb{N}$. Then

$$\langle f, \mathfrak{z}^n \rangle = \frac{\pi}{n+1} \cdot a_n.$$

PROOF.

$$\begin{aligned}
\langle f, \mathfrak{z}^n \rangle &= \int_\Delta f(z) \overline{z}^n \, dz \\
&= \int_0^1 \rho \int_0^{2\pi} f(\rho e^{i\theta}) (\overline{\rho e^{i\theta}})^n \, d\theta \, d\rho \\
&= \int_0^1 \rho \int_0^{2\pi} \left(\sum_{j=0}^\infty a_j (\rho e^{i\theta})^j \right) (\overline{\rho e^{i\theta}})^n \, d\theta \, d\rho \\
&= \pi \cdot a_n \int_0^1 2\rho \cdot \rho^{2n} \, d\rho \\
&= \frac{\pi}{n+1} \cdot a_n.
\end{aligned}$$

□

The difficulty with the above theorem is that the power series for a function converges locally uniformly. We do not know *a priori* whether the power series also converges with the $\mathcal{L}^2$-norm. Of course, if it did, the proof would simply be an application of the preceding lemma.

COROLLARY 5.3. *Let $f \in A(\Delta, \lambda)$ be such that*

$$f \perp \mathfrak{z}^n$$

for each $n \in \mathbb{N}$. Then $f = 0$.

PROOF. The function f has a power series expansion on the unit disc: $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$. Then $\langle f, \mathfrak{z}^n \rangle = 0$ for all $n \in \mathbb{N}$. But that means that $a_n = 0$ for all $n \in \mathbb{N}$. In other words, $f = 0$. □

COROLLARY 5.4. *Let*

$$e_n(z) = \sqrt{\frac{n+1}{\pi}} \cdot z^n \qquad (z \in \Delta, n \in \mathbb{N})$$

Then $(e_n)_{n \in \mathbb{N}}$ is an orthonormal base for the Bergman space $A(\Delta, \lambda)$.

PROOF. We have just seen that $\|e_n\| = 1$ for all $n \in \mathbb{N}$. Let $f \in A(\Delta, \lambda)$ be such that $f \perp e_n$ for all n. Then also $f \perp \mathfrak{z}^n$ for all n. Hence $f = 0$. □

Note that each function e_n has a norm of 1, but that $e_n \to 0$ locally uniformly!

THEOREM 5.5. *Let K be the Bergman kernel associated with the unit disc Δ and the Lebesgue measure on it. Let $w, z \in \Delta$. Then*

$$K(w, z) = \frac{1}{\pi(1 - w\overline{z})^2}.$$

PROOF.

$$\begin{aligned} K(w, z) &= \sum_{n=0}^{\infty} \sqrt{\frac{n+1}{\pi}} \cdot w^n \cdot \sqrt{\frac{n+1}{\pi}} \cdot \overline{z}^n \\ &= \frac{1}{\pi} \sum_{n=0}^{\infty} (n+1)(w\overline{z})^n \\ &= \frac{1}{\pi(1 - w\overline{z})^2}. \end{aligned}$$

□

COROLLARY 5.6 (Bergman's formula). *Let $f \in H(\Delta)$ be such that*

$$\int_{\Delta} |f(w)|^2 \, dw < \infty.$$

Let $z \in \Delta$. Then

$$f(z) = \frac{1}{\pi} \int_{\Delta} \frac{f(w)}{(1 - \overline{w}z)^2} \, dw.$$

CHAPTER 6

Rotation-invariant Measures on the Unit Disc

In this chapter we characterize a certain class of Bergman measures. One thing one might ask about Bergman measures is whether there is any 'geometric' property that characterizes them. For example, every Bergman measure μ on Ω must satisfy $\mu(\Omega \setminus E) > 0$ for every compact subset E of Ω. This property, although necessary, is far from sufficient. However, if we restrict ourselves to a certain class of measures the property *does* become sufficient.

We define a map T from the set of measures on the 'unit interval' $[\,0,1)$ to the set of measures on the unit disc Δ as follows.

Definition. Let ν be a measure on $[\,0,1)$. For a measurable subset X of Δ, let

$$T(\nu)(X) = \frac{1}{2}\int_{[\,0,1)}\int_0^{2\pi} \mathbf{1}_X(\sqrt{\rho}e^{i\theta})\,d\theta\,d\nu(\rho).$$

For any measurable function f on the unit disc for which $\int_\Delta |f|\,dT(\nu) < \infty$, we have

$$\int_\Delta f\,dT(\nu) = \frac{1}{2}\int_{[\,0,1)}\int_0^{2\pi} f(\sqrt{\rho}e^{i\theta})\,d\theta\,d\nu(\rho).$$

Using polar coordinates, one quickly verifies that

$$T\big(\lambda_{[\,0,1)}\big) = \lambda_\Delta.$$

More generally, let $\omega\colon [\,0,1) \to [\,0,\infty\,]$ be a measurable function. Define the function $\tilde{\omega}\colon \Delta \to [\,0,\infty\,]$ by

$$\tilde{\omega}(z) = \omega\big(|z|^2\big).$$

Then

$$T\big(\omega\lambda_{[\,0,1)}\big) = \tilde{\omega}\lambda_\Delta.$$

Note that $T(\nu)(\Delta_r) = \pi\cdot\nu\big([\,0,r^2)\big)$ for all $r \in (0,1\,]$. In particular, if the measure ν is finite, then so is $T(\nu)$, and vice versa.

Theorem 6.1. *Let ν be a measure on $[\,0,1)$ that is such that $T(\nu)$ is a Bergman measure. Then*

$$\nu\big([0,r)\big) < \infty \quad \textit{and} \quad \nu\big([\,r,1)\big) > 0 \qquad \textit{for each } r \in (0,1).$$

Proof. Let $r \in (0,1)$. Choose a number $s \in (0,1)$ such that $s^2 > r$. The set Δ_s is contained within a compact subset of Δ, and hence $T(\nu)(\Delta_s) < \infty$. It follows that $\nu\big([\,0,r)\big) < \infty$. On the other hand, the set $\Delta \setminus \overline{\Delta}_s$ cannot have measure zero, by theorem 3.2. It then follows that $\nu\big([\,r,1)\big) > 0$. $\square$

Amazingly, the converse of the above theorem is also true.

Theorem 6.2. *Let ν be a measure on $[\,0,1)$ that is such that*

$$\nu\big([0,r)\big) < \infty \quad \textit{and} \quad \nu\big([\,r,1)\big) > 0 \qquad \textit{for each } r \in (0,1).$$

Then $T(\nu)$ is a Bergman measure.

PROOF. Let E be a compact subset of Δ. Then there exists a number $r \in (0, 1)$ such that $E \subseteq \Delta_r$. Since $T(\nu)(\Delta_r) = \pi \cdot \nu\big([\,0, r^2)\big) < \infty$, it follows that $T(\nu)(E) < \infty$.

Define numbers $M_0,\ M_1,\ M_2,\ \ldots$ in $[\,0, \infty\,]$ as follows:

$$M_n = \int_{[\,0,1)} t^n \, d\nu(t) \qquad (n \in \mathbb{N}).$$

Let $z \in \Delta$. There exists a number $r \in [\,0, 1)$ be such that $r > |z|$. Choose such a number. Let $n \in \mathbb{N}$. Then

$$\int_{[\,0,1)} t^n \, d\nu(t) \geq \int_{[\,r,1)} t^n \, d\nu(t) \geq \int_{[\,r,1)} r^n \, d\nu(t) = r^n \cdot \nu\big([r, 1)\big).$$

We see that

$$M_n > 0$$

and

$$\left| \frac{1}{M_n} z^n \right| \leq \frac{1}{\nu\big([r,1)\big)} \cdot \left| \frac{z}{r} \right|^n .$$

It follows that the power sequence

$$\sum_{n=0}^{\infty} \frac{1}{M_n} z^n$$

converges for every $z \in \Delta$. (Note that there may be some numbers n for which M_n is infinitely large.)

Define a function $K \colon \Delta \times \Delta \to \mathbb{C}$ as follows:

$$K(w, z) = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{1}{M_n} (w\overline{z})^n \qquad (w, z \in \Delta).$$

Let $f, g \in H(\Delta)$ be such that $\int_\Delta |f|^2 \, dT(\nu) < \infty$ and $\int_\Delta |g|^2 \, dT(\nu) < \infty$. Choose numbers $a_0,\ a_1,\ a_2,\ \ldots$ and $b_0,\ b_1,\ b_2,\ \ldots$ such that, for each $z \in \Delta$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$.

We try to express the norm of f as well as the inner product of f and g in terms of the numbers a_n and b_n.

We have

$$\begin{aligned}
\int_\Delta |f|^2 \, dT(\nu) &= \frac{1}{2} \int_{[\,0,1)} \int_0^{2\pi} \big| f(\sqrt{\rho} e^{i\theta}) \big|^2 \, d\theta \, d\nu(\rho) \\
&= \pi \int_{[\,0,1)} \sum_{n=0}^{\infty} |a_n|^2 \rho^n \, d\nu(\rho) \\
&= \pi \sum_{n=0}^{\infty} |a_n|^2 M_n.
\end{aligned}$$

(Note that the above also holds in case $\int_\Delta |f|^2 \, dT(\nu) = \infty$.)

Similarly, we see that

$$\int_\Delta f\overline{g} \, dT(\nu) = \pi \sum_{n=0}^{\infty} a_n \overline{b_n} M_n.$$

Let $z \in \Delta$. Denote the function $w \mapsto K(w, z)$ by K_z. Obviously, $K_z \in H(\Delta)$. But we also see that

$$\int_\Delta |K_z|^2 \, dT(\nu) = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{|z|^2}{M_n} = K(z, z).$$

In particular, $\int_\Delta |K_z|^2\, dT(\nu) < \infty$. It follows that

$$\int_\Delta f\overline{K_z}\, dT(\nu) = \sum_{n=0}^{\infty} a_n z^n = f(z),$$

hence

$$\bigl|f(z)\bigr|^2 \leq K(z,z) \cdot \int_\Delta |f|^2\, dT(\nu).$$

The function $z \mapsto K(z,z)$ is bounded on compact subsets of Δ. It follows that $T(\nu)$ is a Bergman measure. $\square$

The proof also gives some impression as to what the Bergman space and the Bergman kernel look like. In particular, we have the following.

THEOREM 6.3. *Let ν be a measure on $[0,1)$ such that $T(\nu)$ is a Bergman measure. If ν is finite, the space $A\bigl(\Delta, T(\nu)\bigr)$ is infinite-dimensional, and the functions $\mathfrak{z}^0$, $\mathfrak{z}^1$, $\mathfrak{z}^2$, … form an orthogonal base for that space. If ν is infinite, $A\bigl(\Delta, T(\nu)\bigr)$ consists only of the constant function* 0.

PROOF. Let M_0, M_1, M_2, … and the function K be as in the proof of the previous theorem.

Suppose, ν is infinite. Then the numbers M_n are all infinite. It follows that $K(z,z) = 0$ for every $z \in \Delta$, and hence $A\bigl(\Delta, T(\nu)\bigr) = \{0\}$.

Suppose, ν is finite. Then the (infinitely many) functions $\mathfrak{z}^0$, $\mathfrak{z}^1$, $\mathfrak{z}^2$, … are elements of $A\bigl(\Delta, T(\nu)\bigr)$. One quickly verifies that each of these is pairwise orthogonal. Hence, $A\bigl(\Delta, T(\nu)\bigr)$ is infinite-dimensional. Let $f \in A\bigl(\Delta, T(\nu)\bigr)$ be such that $f \perp \mathfrak{z}^n$ for all $n \in \mathbb{N}$. By evaluating $\langle f, \mathfrak{z}^n\rangle$ as in the proof of the previous theorem, we conclude that all power series coefficients of f must vanish, hence $f = 0$. It follows that the functions $\mathfrak{z}^0$, $\mathfrak{z}^1$, $\mathfrak{z}^2$, … form an orthogonal base for the space $A\bigl(\Delta, T(\nu)\bigr)$. $\square$

All measures in the range of T are rotation-invariant, that is, every measure μ in the range of T satisfies

$$\mu(e^{i\theta}X) = \mu(X)$$

for every measurable set $X \subseteq \Delta$ and every real number θ. In fact, *every* rotation-invariant measure on the unit disc is in the range of T. (This is the only reason for introducing T to begin with.)

THEOREM 6.4. *Let μ be a rotation-invariant measure on the unit disc. Then there exists a measure ν on $[0,1)$ such that $\mu = T(\nu)$.*

PROOF. Let φ be the function on the unit disc defined by $\varphi(z) = |z|^2$. Define a measure ν on $[0,1)$ by

$$\nu(Y) = \frac{1}{\pi}\mu\bigl(\varphi^{-1}(Y)\bigr).$$

Let $\mu' = T(\nu)$. Then μ' is a rotation-invariant measure on the unit disc. We are done if we can show that the measures μ and μ' are identical. Let $a, b \in [0,1)$, $a < b$. By definition of ν,

$$\mu\bigl(\varphi^{-1}([a,b))\bigr) = \pi \cdot \nu\bigl([a,b)\bigr).$$

And by definition of μ',

$$\begin{aligned}\mu'\bigl(\varphi^{-1}([a,b))\bigr) &= \frac{1}{2}\int\int_0^{2\pi} \mathbf{1}_{\varphi^{-1}([a,b))}(\sqrt{\rho}e^{i\theta})\, d\theta\, d\nu(\rho)\\ &= \pi\int \mathbf{1}_{\varphi^{-1}([a,b))}(\sqrt{\rho})\, d\nu(\rho)\\ &= \pi \cdot \nu\bigl[a,b)\bigr).\end{aligned}$$

Hence, $\mu(X) = \mu'(X)$ for each set X that is of the form $\varphi^{-1}\big([\,a, b)\big)$, that is, either a disc or an annulus with centre 0. Since both μ and μ' are rotation-invariant, it follows that $\mu(X) = \mu'(X)$ for every set X that is of the form

$$\big\{z \in \Delta : a \leq |z|^2 < b \text{ and } \theta - \tfrac{\pi}{n} \leq \arg z < \theta + \tfrac{\pi}{n}\big\},$$

where $a, b \in [\,0, 1)$, $\theta \in \mathbb{R}$ and $n \in \mathbb{N}^*$. But the Borel-σ-algebra of Δ is generated by these sets. It follows that $\mu = \mu'$. $\square$

We can now completely characterize all rotation-invariant Bergman measures on the unit disc.

THEOREM 6.5. *Let μ be a rotation-invariant measure on the unit disc. Then μ is a Bergman measure if any only if*

$$\mu(\Delta_r) < \infty \quad \textit{and} \quad \mu(\Delta \setminus \Delta_r) > 0 \qquad \textit{for every } r \in (0, 1).$$

THEOREM 6.6. *Let μ be a rotation-invariant Bergman measure on the unit disc. If μ is finite, the space $A(\Delta, \mu)$ is infinite-dimensional. Otherwise, $A(\Delta, \mu) = \{0\}$.*

CHAPTER 7

The Bergman Kernel and Conformal Mappings

We conclude this paper with the classical result linking the Bergman kernel to the problem of mapping simply-connected domains onto the unit disc.

THEOREM 7.1. *Let Ω_1 and Ω_2 be two nonempty open subsets of $\mathbb{C}$ that are such that there exists a conformal mapping φ from Ω_1 onto Ω_2. Then the Bergman spaces $A(\Omega_1, \lambda_{\Omega_1})$ and $A(\Omega_2, \lambda_{\Omega_2})$ are isomorphic.*

PROOF. During the proof, the inner products on the spaces $A(\Omega_1, \lambda_{\Omega_1})$ and $A(\Omega_2, \lambda_{\Omega_2})$ will be denoted by $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$, respectively.

Let $f \in H(\Omega_2)$. By the transformation theorem,

$$\int_{\Omega_2} |f(w)|^2\, dw = \int_{\Omega_1} \big|f\big(\varphi(w)\big)\big|^2 \cdot \big|\varphi'(w)\big|^2\, dw.$$

It follows that

$$f \in A(\Omega_2, \lambda_{\Omega_2}) \iff f \circ \varphi \cdot \varphi' \in A(\Omega_1, \lambda_{\Omega_1}).$$

Hence, we can define a linear operator T from $A(\Omega_2, \lambda_{\Omega_2})$ to $A(\Omega_1, \lambda_{\Omega_1})$ by

$$Tf = f \circ \varphi \cdot \varphi' \qquad (f \in A(\Omega_2, \lambda_{\Omega_2}).)$$

Let $g, h \in A(\Omega_2, \lambda_{\Omega_2})$. Then

$$\begin{aligned}
\langle Tg, Th\rangle_1 &= \int_{\Omega_1} g\big(\varphi(w)\big) \cdot \varphi'(w) \cdot \overline{h\big(\varphi(w)\big) \cdot \varphi'(w)}\, dw \\
&= \int_{\Omega_1} g\big(\varphi(w)\big) \cdot \overline{h\big(\varphi(w)\big)} \cdot \big|\varphi'(w)\big|^2\, dw \\
&= \int_{\Omega_2} g(w) \cdot \overline{h(w)}\, dw \\
&= \langle g, h\rangle_2
\end{aligned}$$

It follows that T is an isometry.

Let ψ be the inverse of φ. Then ψ is a conformal mapping from Ω_2 onto Ω_1. According to the chain rule for differentiation,

$$\psi' \circ \varphi \cdot \varphi' = 1,$$

and

$$\varphi' \circ \psi \cdot \psi' = 1.$$

It is then easy to see that the operator

$$f \mapsto f \circ \psi \cdot \psi'$$

is an inverse of T. Hence, T is unitary. □

THEOREM 7.2. *Let Ω_1 and Ω_2 be two nonempty open subsets of $\mathbb{C}$ that are such that there exists a conformal mapping φ from Ω_1 onto Ω_2. Let K_1 be the Bergman kernel of $A(\Omega_1, \lambda_{\Omega_1})$, and let K_2 be the Bergman kernel of $A(\Omega_2, \lambda_{\Omega_2})$. Then*

$$K_1(w, z) = K_2\big(\varphi(w), \varphi(z)\big) \cdot \varphi'(w) \cdot \overline{\varphi'(z)} \qquad \textit{for all } w, z \in \Omega_1.$$

PROOF. Let T be the unitary operator that we used in the proof of the previous theorem:

$$Tf = f \circ \varphi \cdot \varphi' \qquad (f \in A(\Omega_2, \lambda_{\Omega_2}).)$$

Let $z \in \Omega_1$.

Let $f \in A(\Omega_2, \lambda_{\Omega_2})$. Then

$$Tf(z) = \langle Tf, K_{1z} \rangle_1.$$

But also

$$\begin{aligned} Tf(z) &= \varphi'(z) \cdot f\big(\varphi(z)\big) \\ &= \varphi'(z) \cdot \langle f, K_{2\varphi(z)} \rangle_2 \\ &= \varphi'(z) \cdot \langle Tf, TK_{2\varphi(z)} \rangle_1 \\ &= \langle Tf, \overline{\varphi'(z)} \cdot TK_{2\varphi(z)} \rangle_1 \end{aligned}$$

Summarizing, we have

$$\langle Tf, K_{1z} \rangle_1 = \langle Tf, \overline{\varphi'(z)} \cdot TK_{2\varphi(z)} \rangle_1 \qquad \text{for all } f \in A(\Omega_2, \lambda_{\Omega_2}).$$

Since T is surjective, it follows that

$$\begin{aligned} K_{1z} &= \overline{\varphi'(z)} \cdot TK_{2\varphi(z)} \\ &= \overline{\varphi'(z)} \cdot K_{2\varphi(z)} \circ \varphi \cdot \varphi' \end{aligned}$$

□

COROLLARY 7.3. *Let Ω be an open simply-connected proper subset of $\mathbb{C}$. Let $z \in \Omega$. There exists a unique conformal mapping φ that maps Ω onto the unit disc and that is such that $\varphi(z) = 0$ and $\varphi'(z) > 0$. Let K be the Bergman kernel associated with Ω and the Lebesgue measure. Then, for all $w \in \Omega$,*

$$K(w, z) = \frac{1}{\pi} \cdot \varphi'(w) \cdot \varphi'(z),$$

and

$$\varphi'(w) = \sqrt{\frac{\pi}{K(z,z)}} \cdot K(w, z).$$

PROOF. The existence of φ is established by the Riemann mapping theorem. Let K_Δ be the Bergman kernel associated with the unit disc and the Lebesgue measure. Let $w \in \Omega$. Then

$$\begin{aligned} K(w, z) &= K_\Delta\big(\varphi(w), \varphi(z)\big) \cdot \varphi'(w) \cdot \overline{\varphi'(z)} \\ &= \frac{1}{\pi\left(1 - \varphi(w)\overline{\varphi(z)}\right)^2} \cdot \varphi'(w) \cdot \overline{\varphi'(z)} \\ &= \frac{1}{\pi} \cdot \varphi'(w) \cdot \varphi'(z). \end{aligned}$$

This proves the first equation.

Since $\varphi'(z)$ is positive, it follows that $\varphi'(z) = \sqrt{\pi \cdot K(z, z)}$. This proves the second equation. □

For simply-connected regions, one can construct the Bergman kernel from a mapping onto the unit disc. Remarkably, we see that one can also do the opposite, i.e. construct a mapping onto the unit disc using the Bergman kernel. Most of the time however, computing a Bergman kernel proves to be completely impossible, so unfortunately this method of constructing a conformal mapping onto the unit disc is not as useful as it might appear.

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